Available at: http://www.ictp.trieste.it/~pub_off IC/2001/126

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

BOUNDED ELEMENTS IN LOCALLY C*-ALGEBRAS

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Abstract

In order to get more useful information about Locally C*-algebras, we introduce in this paper the notion of bounded elements. First, we study the connection between bounded elements and spectrally bounded elements. Some structural results of Locally C*-algebras are established in Theorems 1 , 2 and 3. As an immediate consequence of Theorem 3, we give a characterization of the connected component of the identity in the group of unitary elements for a Locally C*-algebra.

MIRAMARE – TRIESTE
September 2001

1- INTRODUCTION.

Locally C*-algebras have been carefully studied by A. Inoue in [2]. Many results established in the C*-algebras case are extended to the case of locally C*-algebras. Essentially, the similar properties of special elements are established. In this paper, we introduce bounded elements of locally C*-algebras which are useful to have more information about this kind of algebras. In section 2, we study the connection between spectrally bounded elements and bounded elements. In section 3, we give some propositions concerning the stability of bounded elements by *-homomorphisms. The main result in this section is that the C*-algebra of bounded elements of a non trivial locally C*-algebra is not simple. On the other hand, unitary elements are bounded. It is well known that when an algebra $\mathcal{A}$ is C*-algebra, the connected component $\mathcal{U}_0(\mathcal{A})$ of the identity in the group of unitary elements $\mathcal{U}(\mathcal{A})$ is the set of all finite products $e^{ia_1}e^{ia_2}......e^{ia_n}$ with $n \geq 1$ and $a_1, a_2, ..., a_n$ hermitian. In section 3, we show that this set is dense in $\mathcal{U}_0(\mathcal{A})$.

Let $\mathcal{A}$ be a complete topological star algebra with a topology induced by a separating directed set of algebra C*-seminorms $(\| \|_i, i \in I)$. Such an algebra is known as a locally C*-algebra [2]. For each $i \in I$, the set $\ker \| \|_i$ is a closed two-sided ideal of $\mathcal{A}$ and so, $\mathcal{A}/\ker \| \|_i$ is a normed algebra. The completion $\mathcal{A}_i$ of $\mathcal{A}/\ker \| \|_i$ is a C*-algebra and it is called factor algebra associated with $\| \|_i$. The canonical homomorphism π_i from $\mathcal{A}$ into $\mathcal{A}_i$ is *-homomorphism. On the other hand, $\ker \| \|_j \subseteq \ker \| \|_i$ when $j > i$. That induces a canonical homomorphism π_{ij} from $\mathcal{A}_j$ into $\mathcal{A}_i$ such that $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ and $\pi_{ij} \circ \pi_j = \pi_i$ for all $k > j > i$. This directed set of C*-algebras $(\mathcal{A}_i, i \in I)$ together with the collection of continous *-homomorphisms $(\pi_{ij})_{j>i}$, $\pi_{ij} : \mathcal{A}_j \to \mathcal{A}_i$, is said to be a projective system. Endow $\Pi_{i\in I}\mathcal{A}_i$ with the product topology and coordinate-wise operations. Then the subalgebra $LP(\mathcal{A}) = \{ \ (a_i)_{i\in I} : \pi_{ij}(a_j) = a_i \text{ when } i < j \ \}$ is called the inverse limit of $(\mathcal{A}_i)_{i\in I}$.

Proposition 1 [2] *Let $\mathcal{A}$ be a locally C*-algebra with the family of C*-seminorms $(\| \|_i, i \in I)$ and let $(\mathcal{A}_i, \pi_{ij}, i \in I)$ and $LP(A)$ be as above. Then,*

(1) $\mathcal{A}$ is isomorphic to $LP(\mathcal{A})$ and for each $(a)_{i\in I} \in LP(\mathcal{A})$, there exists only one a in $\mathcal{A}$ such that $\pi_i(a) = a_i$.

(2) An element $a \in \mathcal{A}$ is invertible if, and only if, $\pi_i(a)$ is invertible in $\mathcal{A}_i$ for all i in I.

We write $\mathcal{A} \cong \lim_{\longleftarrow} \ (\mathcal{A}_i, \ \pi_{ij})_{i\in I}$. A locally C*-algebra $\mathcal{A}$ is metrizable if its topology is defined by a countable set of C*-seminorms.

2. SPECTRALLY BOUNDED ELEMENTS.

Let $\mathcal{A}$ be a locally C*-algebra with identity. The spectrum $Sp_{\mathcal{A}}(a)$ of an element a in $\mathcal{A}$ is the set of all complex scalars λ such that $\lambda - a$ is not invertible and the spectral radius of a is $\rho_{\mathcal{A}}(a) = Sup\{|\lambda|, \lambda \in Sp_A(a)\}$. With notations as above, we have

$$Sp_{\mathcal{A}}(a) = \cup_{i \in I} Sp_{\mathcal{A}_i}(a_i) \quad for \ all \ a \in \ \mathcal{A}. \tag{1}$$

An element a of$A$ is spectrally bounded in $\mathcal{A}$ if

$$\rho_{\mathcal{A}}(a) = Sup\{\rho_{\mathcal{A}_i}(a_i), i \in I\} < \infty. \tag{2}$$

Consider the set:

$$b(\mathcal{A}) = \{a \in \mathcal{A} : \|a\|_\infty = Sup\{\|a\|_i, i \in I\} < +\infty\}$$

then $(b(\mathcal{A}), \| \ \|_\infty)$ is a C*-algebra. It is called the C*-algebra of bounded elements of $\mathcal{A}$. Such an algebra is dense in $\mathcal{A}$ and the canonical *-homomorphisms $\pi_i : \mathcal{A} \to \mathcal{A}_i$ and $\pi_i : b(\mathcal{A}) \to \mathcal{A}_i$ are surjective [5].

Remark 1. Let $a \in \mathcal{A}$. If $a \in b(\mathcal{A})$, then $Sp_A(a) \subseteq Sp_{b(\mathcal{A})}$ and so, a is spectrally bounded. The converse is not true. Indeed, consider the infinite matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & . & . & . & . & . & . & . \\ 0 & 0 & 2 & 0. & . & . & . & . & . & . \\ . & . & 0 & 3 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & n & 0 & . & \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

Then N is a quasinilpotent element of the locally C*-algebra $\Pi_{n \geq 1} IM_n(\mathbb{C})$ and

$$\|N\|_\infty \geq \sqrt{\rho(\pi_n(N)\pi_n(N)^*} \geq n - 1 \quad for \quad all \quad n \geq 1$$

where π_n is the canonical homomorphism from $\Pi_{n \geq 1} IM_n(\mathbb{C})$ to $IM_n(\mathbb{C})$.

Proposition 2. *Every spectrally bounded hermitian element of a locally C*-algebra $\mathcal{A}$ is in $b(\mathcal{A})$.*

Proof: Using (2) , we have $\|a\|_\infty = \rho_{\mathcal{A}}(a) < \infty$ and so $a \in b(\mathcal{A})$.

A locally C*-algebra is called Q-algebra if its group of invertible elements is an open set. In such algebra, every element is spectrally bounded.

Theorem 1. *Let $\mathcal{A}$ be a metrizable locally C*-algebra with an identity. If every a of $\mathcal{A}$ is spectrally bounded, then $\mathcal{A}$ is C*-algebra. In particular, if A is Q-algebra, then $\mathcal{A}$ is C*-algebra.*

Proof: Assume that $Sp_{\mathcal{A}}(a)$ is bounded for all $a \in \mathcal{A}$. By Proposition 2, all hermitian elements are in $b(\mathcal{A})$ and so, $\mathcal{A} = b(\mathcal{A})$. It remains to show that the linear map $id : \mathcal{A} \to b(\mathcal{A})$ is continuous. Let a_n be a sequence of $\mathcal{A}$ which converges to 0 in $\mathcal{A}$ and converges to a in $b(\mathcal{A})$. Let p be an arbitrary positive linear map on $b(\mathcal{A})$. It is a positive linear map on $\mathcal{A}$. Notice that $\mathcal{A}$ is a metrizable topological algebra with identity. Applying Theorem 3 in [4], p is continuous on $\mathcal{A}$ and so, $p(a_n) \to p(a) = 0$. Therefore, $a = 0$.

Generally, this result is not true. Indeed, consider the $C_c[0,1]$ of continuous complex fonctions on $[0,1]$ with the topology of uniform convergence on countable compact sets. It is easy to see that $C_c[0,1]$ is a locally C*-algebra with spectrally bounded elements, but it is not a C*-algebra.

3. THE C*-ALGEBRA OF BOUNDED ELEMENTS.

In general, we can define $b(\mathcal{A})$ of a complete locally multiplicatively convex algebra $b(\mathcal{A})$ for each family of sub-multiplicative seminorms which induce the topology. Notice that such family of seminorms is not unique. By the following proposition, we deduce that $b(\mathcal{A})$ does not depend on the choice of the family of C*-seminorms which define the topology of $\mathcal{A}$.

Proposition 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two locally C*-algebras.*
1. If $\mathcal{A}$ is a closed star subalgebra of $\mathcal{B}$, then $b(\mathcal{A}) = b(\mathcal{B}) \cap \mathcal{A}$.
*2.If $\phi : \mathcal{A} \to \mathcal{B}$ is an injective *-homomorphism from $\mathcal{A}$ into $\mathcal{B}$ with a closed image, then $b(\mathcal{A}) = b(\mathcal{B}) \cap \phi(\mathcal{A})$.*

Proof: 1. Let a be a hermitian element in $b(\mathcal{A})$. As $\mathcal{A}$ is a closed star subalgebra of $\mathcal{B}$, $Sp_{\mathcal{A}}(a) = Sp_{\mathcal{B}}(a)$. Moreover, $Sp_{\mathcal{A}}(a) \subseteq Sp_{b(\mathcal{A})}(a)$. Then a is spectrally bounded in $\mathcal{B}$. Using Proposition 2, we obtain $a \in b(\mathcal{B})$. Conversely, let a be a hermitian element in $\mathcal{A} \cap b(\mathcal{A})$. Then $Sp_{\mathcal{A}}(a)$ is bounded. Because $Sp_{\mathcal{A}}(a) = Sp_{\mathcal{B}}(a)$, a is spectrally bounded in $\mathcal{A}$ and hence, $a \in b(\mathcal{A})$.
2. Note that $\phi(\mathcal{A})$ is a closed star subalgebra of $\mathcal{B}$. By 1, we deduce that $b(\phi(\mathcal{A})) = \phi(\mathcal{A}) \cap b(\mathcal{B})$. It remains to prove that $b(\phi(\mathcal{A})) \cong b(\mathcal{A})$. Let b a hermitian element of $b(\phi(\mathcal{A}))$. Then there exists a hermitian element a of $\mathcal{A}$ such that $\phi(a) = b$. Since ϕ is a C*-isomorphism from $\mathcal{A}$ onto $\phi(\mathcal{A})$, $Sp_{\mathcal{A}}(a) = Sp_{\phi(\mathcal{A})}(b)$. Therefore, $a \in b(\mathcal{A})$. Hence, ϕ is a C*-isomorphism from $b(\mathcal{A})$ onto to $b(\phi(\mathcal{A}))$. This completes the proof.

In [2], it was shown that every closed two-sided ideal of a locally C*-algebra is a star subalgebra and every closed star subalgebra of a locally C*-algebra is a locally C*-algebra.

Proposition 4. *Let $\mathcal{J}$ be a closed two-sided ideal of a locally C*-algebra $\mathcal{A}$, then $b(\mathcal{J})$ is a closed two-sided ideal of $b(\mathcal{A})$. Moreover, if $\mathcal{A}/\mathcal{J}$ is complete (to be a Locally C*-algebra) , the sequence*

$$0 \longrightarrow b(\mathcal{J}) \xrightarrow{\alpha} b(\mathcal{A}) \xrightarrow{\beta} b(\mathcal{A}/\mathcal{J}) \longrightarrow 0$$

is exact where α is the identity map and β the canonical projection.

Proof: Using Proposition 3, we have $b(\mathcal{J}) = b(\mathcal{A}) \cap \mathcal{J}$. Therefore, it is clear that $b(\mathcal{J})$ is a two-sided ideal of $b(\mathcal{A})$. Because $b(\mathcal{J})$ is a C*-algebra that is embedded in the C*-algebra $b(\mathcal{A})$, $b(\mathcal{J})$ is closed in $b(\mathcal{A})$. On the other hand, it is easy to see that $Im\alpha \subseteq \ker\beta$. Let $a \in b(\mathcal{A})$ such that $\beta(a) = 0$. Then $a \in \mathcal{J}$ and hence $a \in b(\mathcal{J})$.

Corollary 1. *Let $\mathcal{A}$ be a locally C* algebra with a projective system of C*-algebras $(\mathcal{A}_i, \|\|_i, i \in I)$. Then*

$$0 \longrightarrow b(\ker \|\ \|_i) \xrightarrow{\alpha} b(\mathcal{A}) \xrightarrow{\beta} \mathcal{A}_i \longrightarrow 0$$

is exact for each $i \in I$ and hence, $b(\mathcal{A})/b(\ker \|\ \|_i) \cong \mathcal{A}_i$.

Theorem 2. *Let $\mathcal{A}$ be a locally C*-algebra. If $b(\mathcal{A})$ is simple, then $\mathcal{A}$ is a C*-algebra.*

Proof: Note that a locally C*-algebra which is not a C*-algebra, can never be simple and hence $b(\mathcal{A})$ cannot be simple. Indeed, assume that $(\|\ \|_i, i \in I)$ is the set of C*-seminorms which define the topology of $\mathcal{A}$. If $\mathcal{A}$ is not a C*-algebra, then there is $i_0 \in I$ such that $\ker \|\ \|_{i_0}$ is a proper closed two sided-ideal of $\mathcal{A}$. On the other hand, $b(\ker \|\ \|_{i_0})$ is a closed two-sided ideal of $b(\mathcal{A})$ and $b(\ker \|\ \|_{i_0})$ is dense in $\ker \|\ \|_{i_0}$. Therefore $b(\mathcal{A})$ is not simple.

Problem. Let $\mathcal{A}$ denote a C*-algebra which is not simple. Is there a non trivial locally C*-algebra $\mathcal{B}$ such that $\mathcal{A}$ is the C*-algebra of bounded elements of $\mathcal{B}$?.

4. UNITARY ELEMENTS IN A LOCALLY C*-ALGEBRA.

Let $\mathcal{A}$ be a locally C*-algebra with an identity. Denote by $\mathcal{U}(\mathcal{A})$ the group of unitary elements of $\mathcal{A}$, and by $\mathcal{U}_0$ $(\mathcal{A})$ the connected component of the identity in $\mathcal{U}(\mathcal{A})$. Write $\mathcal{E}(\mathcal{A}) = \{e^{ia_1}e^{ia_2}......e^{ia_n}$, a_k is hermitian, $1 \leq k \leq n$ $\}$. Notice that $\mathcal{U}(\mathcal{A}) = \mathcal{U}$ $(b(\mathcal{A}))$ and $\mathcal{U}_0$ $(b(\mathcal{A})) \subseteq \mathcal{U}(\mathcal{A})_{,}$. Generally, $\mathcal{U}_0$ $(\mathcal{A})$ contains elements not in $\mathcal{U}_0$ $(b(\mathcal{A}))$. If not then, as $\mathcal{E}$ $(b(\mathcal{A})) \subseteq \mathcal{E}(\mathcal{A})$ and $\mathcal{E}$ $(b(\mathcal{A})) = \mathcal{U}_0$ $(b(\mathcal{A}))$, we have $\mathcal{U}_0(\mathcal{A}) = \mathcal{E}(\mathcal{A})$. This is not possible in general, since $\mathcal{E}(\mathcal{A})$ need not be closed (see Example 3. 7 in [6]).

Lemma 1. *If $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is an *-homomorphism from a locally C^*-algebra $\mathcal{A}$ onto a locally C^*-algebra $\mathcal{B}$, then $\theta(\mathcal{E}(\mathcal{A})) = \mathcal{E}(\mathcal{B})$.*

Proof: Let $\Pi_{1\leq k\leq n}e^{ik} \in \mathcal{E}(\mathcal{A})$. As θ is an *-homomorphism $\theta(\mathcal{A}) = \mathcal{B}$, there exist hermitian elements $a_k, 1 \leq k \leq n$ in $\mathcal{A}$ such that $\theta(a_k) = b_k$. Therefore $\theta(\Pi_{1\leq k\leq n}e^{ia_k}) = \Pi_{1\leq k\leq n}e^{ib_k}$.

Lemma 2. *Let X be a complete metrizable locally convex linear space and Y a closed subspace of X. If X is the countable inverse limit of a projective system of Banach spaces $(X_i, \pi_i, i \in I)$, then Y is the inverse limit of the projective system $(\overline{\pi_i(Y)}, \pi_i, i \in I)$.*

Proof: See Lemma 2 in [1].

Using Lemmas 1 and 2, we obtain these results.

Theorem 3. *Let $\mathcal{A} \cong \lim_{\leftarrow} (\mathcal{A}_i,\ \pi_{ij})_{i\in I}$ be a unital metrizable locally C^*-algebra with the topology induced by a countable set of submultiplicative C^*-seminorms $(\| \ \|_i, i \in I)$. Then*
1. $\mathcal{U}(\mathcal{A}) \cong \lim_{\leftarrow} (\mathcal{U}(\mathcal{A}_i),\ \pi_{ij})_{i\in I})$.
2. $\mathcal{U}_0\ (\mathcal{A}) \cong \lim_{\leftarrow} (\mathcal{U}_0\ (\mathcal{A}_i),\ \pi_{ij})_{i\in I})$.

Proof: Remark that u is unitary if and only if $\pi_i(u)$ is unitary for all $i \in I$. This shows the first result. Second, note that $\mathcal{U}_0\ (\mathcal{A})$ and $\mathcal{U}_0\ (\mathcal{A}_i)$ are respectively closed in $\mathcal{U}(\mathcal{A})$ and $\mathcal{U}_0(\mathcal{A}_i)$. Using Lemma 2, we have $\mathcal{U}_0\ (\mathcal{A}) \cong \lim_{\leftarrow} \overline{\pi_i(\mathcal{U}_0(\mathcal{A}_i))},\ \pi_{ij})_{i\in I}$. And by Lemma 1, we obtain

$$\pi_i(\mathcal{E}(\mathcal{A})) = \mathcal{E}(\mathcal{A}_i) = \mathcal{U}(\mathcal{A}_i)$$

for all $i \in I$. On the other hand we have,

$$\pi_i(\mathcal{E}(\mathcal{A})) \subseteq \pi_i(\mathcal{U}_0(\mathcal{A})) \subseteq \mathcal{U}_0(\mathcal{A}_i).$$

Therefore, $\overline{\pi_i(\mathcal{U}_0(\mathcal{A}))} = \mathcal{U}_0\ (\mathcal{A}_i)$.

Corollary 2. *The set $\mathcal{E}(\mathcal{A})$ is dense in $\mathcal{U}(\mathcal{A})$.*

Proof: By Lemma 2, we have $\overline{\mathcal{E}(\mathcal{A})}$ is the inverse limit of the projective system $(\overline{\pi_i((\overline{\mathcal{E}(\mathcal{A})})}, \pi_{ij}, i \in I)$ and by Lemma 1, we have

$$\pi_i(\mathcal{E}(\mathcal{A})) = \mathcal{E}(\mathcal{A}_i) = \mathcal{U}_0(\mathcal{A}_i)$$

for all $i \in I$. Moreover,

$$\overline{\pi_i(\overline{\mathcal{E}(\mathcal{A})})} \subseteq \pi_i(\mathcal{U}_0(\mathcal{A})) \subseteq \mathcal{U}_0(\mathcal{A}_i).$$

We deduce that $\overline{\mathcal{E}(\mathcal{A})} = \mathcal{U}_0(\mathcal{A})$.

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